A PRODUCTION INVENTORY MODEL FOR DELAY DEMAND AND DETERIORATION WITH STOCK PRODUCTION RATE AND CONSTANT HOLDING COST

Supported by tetfund

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Received: September 17, 2020 **Accepted:** January 30, 2021

Abstract: This paper discusses a production inventory model for delay demand and deterioration. The production rate is a linear function of the on-hand inventory level. The inventory reduces to the level of buffer stock after production. The paper focuses on obtaining the total optimum average inventory cost, optimum ordering cost, optimum time for a maximum inventory and optimum order quantity. Numerical examples are provided to support our findings.

Keywords: Production, inventory, delayed demand, optimal cost, deterioration and buffer stock

Introduction

In real life situation, demand and inventory level may affect production. When demand for a production decreases (or increases) it makes the manufacturers to decrease (or increase) their production as well. It is observed that the production may be influenced by the demand, which is the production rate may go up or down with the demand rate. This situation generally arises in the case of inventories of highly demandable products. If the demand increases, consumption by the customers obviously will be more and to meet the requirement of the customers, the manufacturers have to increase their production. Similarly, the consumption by the customers is less as the demand of the particular commodity goes down and, in that case, to avoid the inventory, the manufactures will have to cut down their production.

Many researchers have discussed production inventory model under different conditions, by viewing the production rate as a variable (Bhunia & Maiti, 1997) developed two inventory systems. In the first system, the production rate was dependent on the inventory level, while in the second case, the production rate was dependent on the demand. Su & Lin (2011) combined the two models creating a model where production rate is dependent on both inventory level and demand. They assumed that shortages were allowed with complete backlog and exponentially decreasing demand. Urban (1992) formulated and solved inventory models with finite production rate taking various types of demand namely constant or time dependent or stock dependent. For the solution of inventory models with finite rate of replenishment Deb & Chaudhuri (1986) considered the constant rate of demand whereas Goswanmi (1991) took the time dependent demand and Mondal (1989) considered stock dependent demand. Shirajul & Sharifuddin (2016) formulated an inventory model with constant production rate, linear dependent demand and constant holding cost with buffer stock to minimize inventory cost.

In this paper we have considered a production inventory model with the production as a linear function of the on-hand inventory level. We assumed that there is no demand and decay during production and shortages are not allowed. Numerical examples are given to illustrate the model developed and some sensitivity analyses were carried out on the result obtained.

Formulation of the model

In this model, while $t_1 = 0$, the production $\lambda - aI(t)$ starts with Q inventory and the production remain constant for the entire production cycle. The inventory increases at the rate of $\lambda - aI(t)$ during $t = 0$to $t = t_1$. We assumed that there is no demand and decay during the period $t = 0$ to $t = t_1$. With the help of the above information we can formulate the differential equation of the situation as follows:

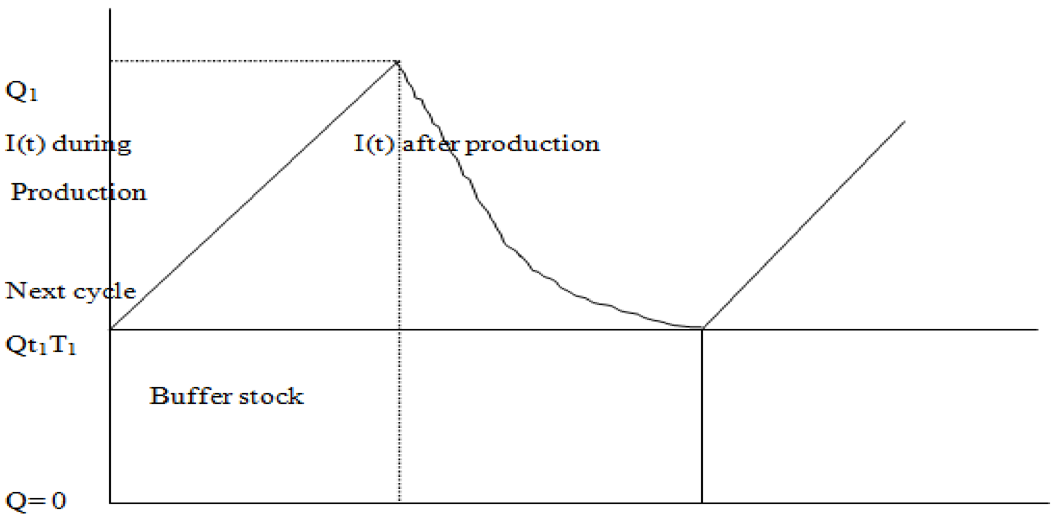

Fig. 1: Inventory before and after production

Assumptions

Production rate $\lambda - aI(t)$ is a linear function of the inventory and greater than demand. The rate of decay μ is small and constant.The demand after production at any instant t is given as$a + bI(t)$, where a and b are constants and satisfying the condition that $\lambda - aI(t) > a + bI(t)$. Production starts with a few items in the inventory as a buffer stock. Inventory level is highest at the end of production$t_1$and after this point the inventory depletes due to demand and deterioration to the level of buffer stock. Shortages are not allowed.

Notations

$I(t)$ = inventory level at time t

R_1 =undecayedinventory for the period from 0 to t_1

R_2 =undecayed inventory for the period from t_1 to T

Q, Q_1 are the inventory levels at time $t = 0$ and $t = t_1$ respectively,Q is the buffer stock.

dt = very small portion of instant t

K_0 = Set up cost.

h =constant holding cost.

$TC = TC(T_1)$ = total average inventory cost in a unit time.

t_1 time when the inventory gets to maximum level.

T_1 = total cycle time.

Q^* = optimum order quantity.

t_1^* optimum time for a maximum inventory.

T_1^* optimum order interval.

$TC(T_1)^*$ optimum average inventory cost per unit time.

$$I(t+dt) = I(t) + \{\lambda - aI(t)\}dt$$

$$I(t+dt) - I(t) = \{\lambda - aI(t)\}dt$$

$$\lim_{dt\to 0}\frac{I(t+dt)-I(t)}{dt} = \lambda - aI(t)$$

$$\frac{dI(t)}{dt} + aI(t) = \lambda$$

$$I(t) = \frac{\lambda}{a} + Ke^{-at} \qquad (1)$$

which is the general solution of the differential equation.Applying the boundary condition

$I(t) = Q$ at $t = 0$

By solving, we get

$$K = Q - \frac{\lambda}{a} \qquad (2)$$

Therefore,

$$I(t) = \frac{\lambda}{a} + \left(Q - \frac{\lambda}{a}\right)e^{-at} \qquad (3)$$

From the other boundary condition i. e. at $t = t_1$, $I(t) = Q_1$, taking up to the first degree of a, we get

$$Q_1 = Q + (\lambda - aQ)t_1 \qquad (4)$$

Using equation (3) and consider the total undecayed in the period $t = 0$ to t_1, taking up to the second degree of a to get

$$R_1 = \int_0^{t_1} I(t)dt = \int_0^{t_1}\left[\frac{\lambda}{a} + \left(Q - \frac{\lambda}{a}\right)e^{-at}\right]dt$$

$$R_1 = \frac{\lambda t}{a} + \left(Q - \frac{\lambda}{a}\right)\frac{e^{-at}}{-a}\Bigg|_0^{t_1}$$

$$R_1 = Qt_1 - \frac{Qat_1^2}{2} + \frac{\lambda t_1^2}{2} \qquad (5)$$

On the other hand, the inventory decreases at the rate of$a + bI(t) + \mu I(t)$during t_1 to T_1, as there is no production after time t_1 and the inventory reduces due to market demand and deterioration.

Applying the same argument as before, we get the differential equations as follows:

$$I(t+dt) = I(t) + \{-a - bI(t) - \mu I(t)\}dt$$

$$I(t+dt) - I(t) = \{-a - bI(t) - \mu I(t)\}dt \qquad (6)$$

$$\lim_{dt\to 0}\frac{I(t+dt)-I(t)}{dt} = -a - bI(t) - \mu I(t)$$

$$\lim_{dt\to 0}\frac{I(t)}{dt} + \mu I(t) + bI(t) = -a$$

$$\lim_{dt\to 0}\frac{I(t)}{dt} + (\mu + b)I(t) = -a$$

$$I(t) = -\frac{a}{(\mu+b)} + Ae^{-(\mu+b)} \qquad (7)$$

which is the general solution of the differential equation.

Applying the boundary condition $t = T_1$ and $I(t) = Q$, by solving we get

$$A = \left(Q + \frac{a}{(\mu+b)}\right)e^{(\mu+b)(T_1-t)}$$

Therefore,

$$I(t) = -\frac{a}{(\mu+b)} + \left(Q + \frac{a}{(\mu+b)}\right)e^{(\mu+b)(T_1-t)} \qquad (8)$$

Now putting the boundary condition $I(t) = Q_1$ when $t = t_1$ taking up to the first degree of μ to obtain

$$Q_1 = Q + (Q(\mu+b) + a)(T_1 - t_1) \qquad (9)$$

Now using equation (7) to get the total undecayed during the period$t = t_1$ to T_1 and considering μ to the first degree.

$$D_1 = \int_{t_1}^{T_1} I(t)dt = \int_{t_1}^{T_1}\left[-\frac{a}{(\mu+b)} + \left(Q + \frac{a}{(\mu+b)}\right)e^{(\mu+b)(T_1-t)}\right]dt$$

$$D_1 = \left[-\frac{at}{(\mu+b)} + \left(Q + \frac{a}{(\mu+b)}\right)\frac{e^{(\mu+b)(T_1-t)}}{-(\mu+b)}\right]\Bigg|_{t_1}^{T_1}$$

$$D_1 = -\frac{a(T_1-t_1)}{2(\mu+b)} + \frac{Q(T_1-t_1)}{2} - \frac{Q(\mu+b)(T_1-t_1)^2}{2} - \frac{a(T_1-t_1)^2}{2} \qquad (10)$$

Now considering the decay during the period $t = t_1$ to T_1 as follows:

$$D_2 = \int_{t_1}^{T_1} \mu I(t)dt = \int_{t_1}^{T_1}\mu\left[-\frac{a}{(\mu+b)} + \left(Q + \frac{a}{(\mu+b)}\right)e^{(\mu+b)(T_1-t)}\right]dt$$

$$D_2 = -\frac{a\mu(T_1-t_1)}{2(\mu+b)} + \frac{Q\mu(T_1-t_1)}{2} - \frac{Q\mu(\mu+b)(T_1-t_1)^2}{2} - \frac{a\mu(T_1-t_1)^2}{2} \qquad (11)$$

From equation (4) and (8) we get

$$t_1 = \frac{[Q(\mu+b)+a]T_1}{\lambda - Qa + Q(\mu+b) + a} \qquad (12)$$

Let

$$V = \frac{Q(\mu+b)+a}{-Qa + Q(\mu+b) + a} \qquad (13)$$

Therefore,

$$t_1 = VT_1 \qquad (14)$$

Now the total cost function is

$$TC(T_1) = \frac{K_0 + h(R_1 + D_1 + D_2)}{T_1} \qquad (15)$$

By using equation (5), (10) and (11) into equation (15) to get

$$TC(T_1) = \frac{1}{T_1}\left\{K_0 + h\left[Qt_1 - \frac{Qat_1^2}{2} + \frac{\lambda at_1^2}{2} - \frac{a(T_1-t_1)}{2(\mu+b)} + \frac{Q(T_1-t_1)}{2} - \frac{Q(\mu+b)(T_1-t_1)^2}{2} - \frac{a(T_1-t_1)^2}{2} - \frac{a\mu(T_1-t_1)}{2(\mu+b)} + \frac{Q\mu(T_1-t_1)}{2} - \frac{Q\mu(\mu+b)(T_1-t_1)^2}{2} - \frac{a\mu(T_1-t_1)^2}{2}\right]\right\}$$

By using equation (14) we get the value of $TC(T_1)$ as

$$TC(T_1) = \frac{K_0}{T_1} + hQV - \frac{ahQV^2T_1}{2} + \frac{ah\lambda V^2T_1}{2} - \frac{ah(1-V)}{2(\mu+b)}(1+\mu) + \frac{hQ(1-V)}{2}(1+\mu) - \frac{hQ(\mu+b)(1-V^2)T_1}{2}(1+\mu) - \frac{ah(1-V^2)T_1}{2}(1+\mu) \qquad (16)$$

The main objective is to find the value of T_1 which gives the minimum variables per unit time. The necessary and sufficient condition to minimize $TC(T_1)$ are respectively;

(i) $\frac{dTC(T_1)}{dT_1} = 0$

and

(ii) $\frac{d^2TC(T_1)}{dT_1^2} > 0$

Therefore, to satisfy the necessary condition we are to differentiate equation (16) with respect to T_1 as follows:

$$\frac{dTC(T_1)}{dT_1} = -\frac{K_0}{T_1^2} - \frac{ahQV^2}{2} + \frac{ah\lambda V^2}{2} - \frac{hQ(\mu+b)(1-V^2)}{2}(1+\mu) - \frac{ah(1-V^2)}{2}(1+\mu) \qquad (17)$$

Equating equation (16) to 0 which minimize the variable cost per unit time

$$\frac{K_0}{T_1^2} = -\frac{ahQV^2}{2} + \frac{ah\lambda V^2}{2} - \frac{hQ(\mu+b)(1-V^2)}{2}(1+\mu) - \frac{ah(1-V^2)}{2}(1+\mu) \qquad (18)$$

Now the optimum order interval is;

$$T_1 = \sqrt{\frac{2K_0}{h[-aQV^2 + a\lambda V^2 - Q(\mu+b)(1-V^2)(1+\mu) - a(1-V^2)(1+\mu)]}} \qquad (19)$$

By using equation (13) and (18) to get the optimum time for maximum inventory as:

$$t_1 = V\sqrt{\frac{2K_0}{h[-aQV^2 + a\lambda V^2 - Q(\mu+b)(1-V^2)(1+\mu) - a(1-V^2)(1+\mu)]}} \qquad (20)$$

Theorem

The cost function $TC(T_1)$ is convex.

Proof: From equation (17),

$$\frac{d^2TC(T_1)}{dT_1^2} = \frac{K_0}{T_1^3} > 0$$

Therefore, the convex property (ii) is satisfied as both K_0 and T_1 are positive. We conclude that the total cost function is convex in T_1. Hence, there is optimal solution in T_1.

In this section, we provide a numerical example to illustrate the developed model. The values of various parameters are as follows:

$K_0 = 100$, $\lambda = 60$, $\alpha = 4$, $Q = 8$, $h = 2$, $\mu = 0.01$, $b = 3$

Substituting these values into equation (4), (15) and (19) gives $Q_1^* = 33.2221$, $TC(T_1^*) = 346.6438$, $T_1^* = 10.54382$ and $t_1^* = 0.900789$

Sensitivity analysis

We study the effects of changes of parameter K_0, λ, μ, a, Q and b on the optimal time for maximum inventory t_1^*, optimal time cycle T_1^*, optimal order quantity Q_1^* and total average inventory cost $TC(T_1^*)$. We perform the sensitivity analysis by changing each of the parameter by 50%, 25%, 10%, 5%, −5%, −25%, and −50% taking one parameter at a time and keeping the remaining constant. The details are shown in Table 1

Table 1

Parameter	% change	t_1^*	T_1^*	Q_1^*	$TC(T_1^*)$
	50%	1.103237	11.668650	38.890630	384.769800
	25%	1.007113	11.148730	36.199160	367.062300
	10%	0.944756	10.798070	34.453160	355.202500
K_0	5%	0.923034	10.673220	33.844960	350.995400
	−5%	0.877981	10.409480	32.583460	342.134900
	−10%	0.854564	10.269720	31.927780	377.454100
	−25%	0.780106	9.812130	29.842980	322.195400
	−50%	0.639540	8.866260	25.834720	290.965500
	50%	1.055022	7.863783	69.191300	97.654970
	25%	0.933992	9.197465	48.161680	185.812900
	10%	0.906323	9.990707	38.814980	269.777100
λ	5%	0.902352	10.263730	35.972900	305.635100
	−5%	0.901598	10.832820	30.539950	393.816800
	−10%	0.904797	11.132840	27.905540	448.64459
	−25%	0.929835	12.125380	20.087860	676.580700
	−50%	1.043161	14.367590	5.913678	1554.265000
Parameter	**% change**	t_1^*	T_1^*	Q_1^*	$TC(T_1^*)$
	50%	0.735491	11.668650	28.593760	566.442300
	25%	0.805690	11.148730	30.559330	453.782400
	10%	0.858869	10.798070	32.048330	388.777900
h	5%	0.879080	10.673220	32.614250	367.584800
	−5%	0.924900	10.409800	33.877320	325.964800
	−10%	0.949515	10.269672	34.586420	305.558800
	−25%	1.040142	9.812130	37.123970	246.105300
	−50%	1.273908	8.866260	43.669430	153.941800
	50%	0.780548	14.044090	21.366570	1393.940000
	25%	0.822683	12.378760	26.453660	750.574000
	10%	0.862810	11.305940	30.197680	483.755000
Q	5%	0.880335	10.930520	31.640840	411.441900
	−5%	0.924967	10.143830	34.979030	288.915400
	−10%	0.953990	9.728004	36.964480	237.833300
	−25%	1.091456	8.344036	45.292400	120.585500
	−50%	2.750872	4.407189	125.038300	30.961210
	50%	0.594517	15.507220	15.134200	2261.480000
	25%	0.697229	13.036840	21.944870	961.219200
	10%	0.798360	11.871870	27.799330	539.017000
a	5%	0.844610	11.066550	30.297700	435.487900
	−5%	0.970976	9.995366	36.740890	270.690800
	−10%	1.062048	9.408969	41.135900	260.144300
	−25%	1.691681	7.124169	68.90053	70.820950
	−50%	-	-	-	-
	50%	0.871434	11.623900	32.400140	551.494600
	25%	0.883176	11.149760	32.728920	451.703400
	10%	0.892782	10.805530	32.997910	389.151100
b	5%	0.896595	10.678270	33.104660	367.960200
	−5%	0.905425	10.401510	33.351910	325.236000
	−10%	0.910578	10.250560	33.496180	303.776400
	−25%	0.930171	9.736455	34.044800	239.591200
	−50%	0.989197	8.588077	35.697500	137.707500
	50%	0.902179	10.539440	33.261000	345.992900
	25%	0.901483	10.541630	33.241530	346.318700
	10%	0.901067	10.542950	33.229860	346.513900
μ	5%	0.900928	10.543380	33.225980	346.578900
	−5%	0.900651	10.544260	33.218210	346.708800
	−10%	0.900512	10.544690	33.214330	346.773700
	−25%	0.900096	10.546000	33.202700	346.968200
	−50%	0.899405	10.548170	33.183340	347.292000

Analyzing the results in the table above, we can summarize the following observations;

(1) With increase in the value of the parameter K_0, the values of t_1^*, T_1^*, Q_1^* and $TC(T_1^*)$ increases. Here K_0 is very sensitive to all the values in the model.

(2) With increase in the value of the parameterλ, the values of$T_1{}^*$ and $TC(T_1{}^*)$ decreases and $Q_1{}^*$ increases while $t_1{}^*$ primarily decreases and then increases. Here λ is highly sensitive to all the values of the parameter in the model.
(3) The values of $t_1{}^*$ and $Q_1{}^*$ decreases while the values of $T_1{}^*$ and $TC(T_1{}^*)$ increases with an increase in the value of the parameter h. Here h is highly sensitive to $T_1{}^*$ and $TC(T_1{}^*)$ and moderately sensitive to $t_1{}^*$ and $Q_1{}^*$
(4) The value of the parameter Q is highly sensitive to $T_1{}^*$ and moderately sensitive to $t_1{}^*$,$Q_1{}^*$and $TC(T_1{}^*)$ because an increase in the value of the parameter Q resulted to an increase in the value of $T_1{}^*$ and decrease in the values of $t_1{}^*$,$Q_1{}^*$and $TC(T_1{}^*)$.
(5) With an increase in the value of the parameter a, there is a corresponding increase in the values of $T_1{}^*$ and $TC(T_1{}^*)$ while $t_1{}^*$ and $Q_1{}^*$ decreases. Here a is highly sensitive to $T_1{}^*$ and $TC(T_1{}^*)$ and moderately sensitive to $t_1{}^*$ and $Q_1{}^*$.
(6) The values of $t_1{}^*$ and $Q_1{}^*$ decreases while the values of $T_1{}^*$ and $TC(T_1{}^*)$ increases with an increase in the value of the parameter b. Here b is moderately sensitive to $t_1{}^*$ and $Q_1{}^*$ and highly sensitive to $T_1{}^*$ and $TC(T_1{}^*)$.
(7) With an increase in the value of the parameter μ, the values of $t_1{}^*$,$Q_1{}^*$ and $TC(T_1{}^*)$ all increases while the value of $T_1{}^*$ decreases. Hereμ is highly sensitive to $t_1{}^*$,$Q_1{}^*$ and $TC(T_1{}^*)$ but moderately sensitive to $T_1{}^*$.

Conclusion

We have developed a production inventory model for delay demand and deterioration with stock production rate. The model helps to determine the optimum order quantity, optimum time for a maximum inventory, optimum order interval and optimum average inventory cost per unit time. Numerical example is presented to illustrate the application of the model and some sensitivity analysis was carried out on the result obtained.

Conflict of Interest

The authors declare that there is no conflict of interest related to this study.

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